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# Quantum time delay in chaotic scattering: a semiclassical approach

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**Abstract.** We study the universal fluctuations of the Wigner–Smith time delay for systems which exhibit chaotic dynamics in their classical limit. We present a new derivation of the semiclassical relation of the quantum time delay to properties of the set of trapped periodic orbits in the repeller. As an application, we calculate the energy correlator in the crossover regime between preserved and fully broken time reversal symmetry. We discuss the range of validity of our results and compare them with the predictions of random matrix theories.

## 1. Introduction

Over the past decade many studies have been devoted to the understanding of quantum manifestations of classical chaos. This interest can be explained by the fact that this subject has applications in many different areas of physics, such as properties of complex systems, fundamental aspects of the correspondence principle, transport in ballistic mesoscopic cavities, etc. Most of the theoretical studies have concentrated on spectral properties of closed systems, accumulating a large body of numerical evidence of universality and some analytical understanding of this fact. Comparatively few studies have heretofore been devoted to open systems, and the scattering problem still lacks some solid fingerprints which serve to clearly distinguish integrable from chaotic classical scattering. For this reason, it is desirable to study an observable which bridges the well-understood quantum aspects of closed chaotic systems and the still unclear features of open ones. The Wigner–Smith time delay [1, 2] is such an object, since it is intimately related to the level (or resonance) density of the system and it is a genuine scattering observable. This study deals with the universal features of the time delay common to all chaotic scattering systems.

The concept of time delay in quantum scattering was first considered by Eisenbud [3] and Wigner [1] in the context of one-channel scattering. Later, Smith [2] extended the previous discussions to the many-channel problem by introducing the lifetime matrix

$$Q_{ab}(E) = -i\hbar \sum_{c=1}^{\Lambda} S_{ac}(E) \frac{d}{dE} S_{cb}^{\dagger}(E) \quad (1)$$

where  $S$  is the standard scattering matrix and the sum runs over all  $\Lambda$  open channels denoted by  $c$ . By averaging over the eigenvalues of  $Q$ , one arrives at the so-called Wigner–Smith time delay

$$\tau(E) = -\frac{i\hbar}{\Lambda} \text{Tr} \left[ S^{\dagger}(E) \frac{d}{dE} S(E) \right] = -\frac{i\hbar}{\Lambda} \frac{d}{dE} \log \det S(E) \quad (2)$$

which is then interpreted as the typical time spent by the particle in the interaction region. Even though this interpretation has some limitations in the case of wavepacket scattering [4], no difficulties arise when the incoming wave can be considered monoenergetic, a common situation, for example, in applications to mesoscopic transport phenomena [5, 6] and microwave cavity experiments [7–9].

In general, one can distinguish two regimes associated with a scattering process: a fast response (corresponding to *direct* processes) and a delayed response related to the formation of a long-lived resonance. In the energy domain, direct processes rule the energy-averaged behaviour of  $\tau(E)$ . Alternatively, strong fluctuations on the scale of the mean resonance spacing  $\Delta$  are associated to quasibound states, and are, in turn, intimately linked to the classical dynamics in the interaction region.

Our analysis deals with a specific model which illustrates very nicely the most important properties of chaotic scattering and is well suited to study the Wigner–Smith time delay. Some steps in our considerations take into account system specific properties. However, our main results can be easily extended to other chaotic scattering potentials. Our model consists of an irregularly shaped cavity (denoted by  $R$  in figure 1) attached to a pipe (corresponding to the region  $L$ ). The boundary between the pipe (or ‘waveguide’) and the cavity is arbitrarily chosen at the entrance of the cavity, at  $x = D$  (the region  $R$  need not necessarily be a billiard). The quantum propagation in the direction parallel to the pipe axis is free. In the transversal direction there are quantized modes  $\phi_c(y)$  of energy  $\epsilon_c$ , defining the scattering channels  $c$ . At  $x = 0$ , with  $D$  chosen to be sufficiently large, the wavefunction  $\psi(x, y; E)$  is expressed as a superposition of propagating modes

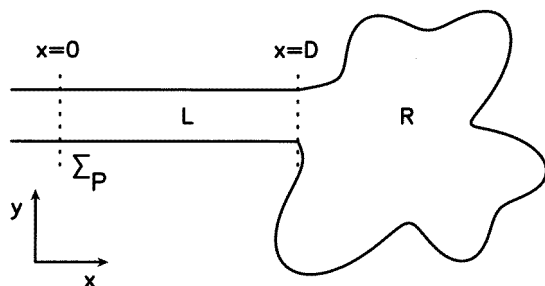
$$\psi(x, y; E) = \sum_{c=1}^{\Lambda} \left( a_c e^{ik_c x} - \sum_{c'=1}^{\Lambda} S_{cc'}(E) a_{c'} e^{-ik_c x} \right) \phi_c(y) \quad (3)$$

with the wavenumbers  $k_c$  given by

$$\frac{\hbar^2}{2m} k_c^2 = E - \epsilon_c. \quad (4)$$

From this equation it becomes evident that by choosing  $|k|D \gg 1$  we ensure that no evanescent mode survives at  $x = 0$  and equation (3) is valid. All the information about the scattering process is contained in the energy-dependent scattering matrix  $S(E)$ .

For chaotic systems, we derive a formula which shows how to calculate the quantum time delay using information about the classical orbits trapped inside the system. We further explore this formula to compute time delay correlators, which show universal features. The universal correlators typically scale with quantities such as the average dwell time inside



**Figure 1.** Schematic illustration of the scattering system under investigation. The ‘cavity’ region is denoted by  $R$  and the attached waveguide by  $L$ .

the system,  $\tau_{\text{dwell}}$ , and the mean resonance spacing  $\Delta$  (also given in terms of the Heisenberg time  $\Delta = 2\pi\hbar/\tau_{\text{H}}$ ). We show that the universal curves obtained in the semiclassical theory are in good agreement with the statistical theory of random matrices when  $\tau_{\text{H}}/\tau_{\text{dwell}} \gg 1$ .

This paper is organised as follows. In section 2, a novel derivation of the quantum time delay in terms of the underlying classical phase space of the repeller is presented. Some applications of this formula are explored in section 3. The comparison with random matrix theory is presented in section 4, where we also discuss the range of validity of the correspondence between semiclassical and statistical theories. In section 5 we present the conclusions of this study.

## 2. Wigner–Smith time delay and trapped periodic orbits

This section is devoted to the derivation of a semiclassical equation for the Wigner–Smith time delay in terms of classical periodic orbits trapped inside the scattering region. Our derivation relies on the association of the  $S$ -matrix with the quantum Poincaré map, following closely the formalism developed by Bogomolny [10] for closed systems. A similar result was previously obtained by Balian and Bloch [11], based on a construction proposed by Friedel [12] for a separable system. The derivation presented below is more transparent than the one in [11], making the approximations more controllable when dealing with actual systems.

We begin by showing a simple construction relating the energy derivatives of two sets of invariants of the  $\Lambda$ -channel scattering matrix  $S$ , namely the eigenphases  $\{\theta_1, \theta_2, \dots, \theta_\Lambda\}$  and the traces  $\{\text{Tr } S^n, n = 1, 2, \dots\}$  (the construction remains valid for an arbitrary unitary matrix depending on one parameter). For this purpose, let us consider the periodic function

$$F(\theta) = \theta \bmod 2\pi \quad (5)$$

which has a Fourier expansion given by

$$F(\theta) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}. \quad (6)$$

Summing over all  $S$ -matrix eigenphases in both sides of equation (6) and using  $\text{Im } \text{Tr } S^n = \sum_{c=1}^{\Lambda} \sin n\theta_c$  we obtain

$$\sum_{c=1}^{\Lambda} F(\theta_c) = \Lambda\pi - 2\text{Im} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr } S^n. \quad (7)$$

The convenience of this arbitrary choice of  $F$  becomes apparent after differentiating all terms in (7) with respect to the energy  $E$ ,

$$\sum_{c=1}^{\Lambda} \frac{\partial \theta_c}{\partial E} = 2\pi \sum_{c=1}^{\Lambda} \frac{\partial \theta_c}{\partial E} \delta(\theta_c \bmod 2\pi) - 2\text{Im} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial \text{Tr } S^n}{\partial E} \quad (8)$$

since by recalling (2) it is easy to identify the l.h.s. of the above expression with the Wigner–Smith time delay. Equation (8) was first obtained by Bogomolny [10] and later rederived by Rouvinez and Smilansky [13]. Those authors were interested in using the transfer (or scattering matrix) approach to develop a quantization procedure for closed systems. The ideas below are quite different, since we are interested in open systems. Indeed, we used the closed system to understand the scattering problem, which is the reverse of the procedure in [13, 14]. To this end, equation (8) only becomes useful after the following steps.

First, we shall consider the scattering matrix  $S$  (at the energy  $E$ ) for the specific system discussed in section 1. Far away from the cavity, at  $x = 0$ , the influence of the evanescent

modes to the wavefunctions is negligible, since  $|k_c|D \gg 1$  (see figure 1). Therefore, according to equation (3), the exact quantization condition for the system closed at  $x = 0$ , becomes  $\det(S - 1) = 0$ , as has already been observed [14]. In other words, one of the eigenphases of the  $S$ -matrix must vanish

$$\theta_c \bmod 2\pi = 0. \quad (9)$$

With this quantization condition, the first term in the r.h.s. of equation (8) is now easily identified with the density of states,  $\rho_{L+R}(E)$ , of the system closed at  $x = 0$ . After proper averaging over some energy interval,  $\rho_{L+R}(E)$  can be decomposed into a smooth and a fluctuating part

$$\sum_{c=1}^{\Lambda} \frac{\partial \theta_c}{\partial E} \delta(\theta_c \bmod 2\pi) = \rho_{L+R}^{\text{av}}(E) + \rho_{L+R}^{\text{fl}}(E) \quad (10)$$

where  $L$  and  $R$  stand for the pipe and cavity regions respectively (see figure 1).

In this context, the matrix  $S$  is interpreted as the *quantum Poincaré map* of the closed system  $L + R$  associated with the section  $\Sigma_P$  ( $x = 0^+$  in figure 1). The construction is quite obvious, but for the sake of completeness let us be explicit: take an asymptotic incoming wavefunction at  $x = 0^+$  and let it be scattered by  $R$ . As a result, one has a matrix that matches the incoming asymptotic waves into the outgoing ones. This defines the quantum return map for  $x = 0^+$  and it is also the definition of the  $S$ -matrix. In order to obtain the Poincaré map, one still needs to describe what happens for  $x < 0^-$ . This, however, is trivial, since  $x < 0^-$  defines the asymptotic region (there is no coupling between channels) and the corresponding Poincaré map is the identity matrix. For the geometry considered here, the reflection by the hard wall is equivalent to the reinjection procedure, defining the so-called Poincaré *scattering* map, originally proposed by Jung [15]. In this way, the  $S$ -matrix can also be viewed as the quantization of the Poincaré scattering map. It is noteworthy that the quantization condition defined by equation (9) has an interesting semiclassical counterpart. For closed systems, the accuracy of the semiclassical quantization procedure based on a Poincaré section requires the section to be traversed by all periodic orbits, such a condition defining a ‘good’ section. If this is not the case, evanescent corrections become essential. By closing the system sufficiently far away from the cavity region we ensure that the evanescent contributions die out, making the exact quantum problem simpler.

Since the  $S$ -matrix can be obtained by the quantization of a classically chaotic map, in the semiclassical approximation its traces can be expressed as a sum over periodic orbits. Actually, the sum over the traces of  $S$  in equation (8) results in an expression very similar to the standard Gutzwiller trace formula [16] for the oscillatory part of the density of states of the system defined by  $L + R$  [13]. The important difference is that in our case the sum is restricted to those periodic trajectories *that touch the section*  $\Sigma_P$ . Decomposing the full set of periodic orbits of the system  $L + R$  into a set that reaches  $\Sigma_P$  and a set which never leaves the cavity  $R$ , one can write

$$\frac{1}{\pi} \text{Im} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial E} \text{Tr} S^n \approx \rho_{L+R}^{\text{fl}}(E) - \rho_R^{\text{fl}}(E) \quad (11)$$

where  $\rho_R^{\text{fl}}(E)$  can be expressed in terms of periodic orbits constricted to the region  $R$ .

Substituting the relations (10) and (11) into equation (8), we obtain

$$\frac{\Lambda}{2\pi\hbar} \tau(E) \approx \rho_{L+R}^{\text{av}}(E) + \rho_{L+R}^{\text{fl}}(E) - (\rho_{L+R}^{\text{fl}}(E) - \rho_R^{\text{fl}}(E)) \quad (12)$$

yielding

$$\tau(E) \approx \frac{2\pi\hbar}{\Lambda}(\rho_{L+R}^{\text{av}}(E) + \rho_R^{\text{fl}}(E)). \tag{13}$$

This is already very close to the expression we are looking for. The problem is that  $\tau$  is measured with respect to  $x = 0$  and we are interested in the time that the particle spends in the cavity region, i.e. the time delay with respect to  $x = D$ . As a consequence, we still have to translate the origin of coordinates to the entrance of the cavity. Under this operation, the  $S$ -matrix transforms as

$$S(x') = e^{-ikD} S(x) e^{-ikD} \tag{14}$$

where  $x' = x - D$  and  $\mathbf{k}$  is a diagonal matrix having the  $k_c$ 's as elements. Taking into account the fact that the time delay is additive with respect to the product of unitary operators, namely,

$$\text{Tr} \left( (S_1 S_2)^\dagger \frac{d}{dE} (S_1 S_2) \right) = \text{Tr} \left( S_1^\dagger \frac{d}{dE} S_1 \right) + \text{Tr} \left( S_2^\dagger \frac{d}{dE} S_2 \right) \tag{15}$$

we arrive at

$$\tau' = \tau - \frac{2\hbar D}{\Lambda} \frac{d}{dE} \sum_{c=1}^{\Lambda} k_c. \tag{16}$$

The second term on the r.h.s. of (16) is a smooth function of energy. It is proportional to the density of states of the region  $L$  (the pipe in figure 1),

$$\frac{2\hbar D}{\Lambda} \frac{d}{dE} \sum_{c=1}^{\Lambda} k_c = \frac{2D}{\Lambda} \sum_{c=1}^{\Lambda} \frac{1}{v_c} = \frac{2\pi\hbar}{\Lambda} \rho_L^{\text{av}}(E). \tag{17}$$

The first identity above says that under a spatial translation by  $D$ , the time delay varies by the classical time it takes to travel to the point displaced by  $D$  and back, averaged over the channels. The second identity is obtained by using the Weyl formula for the quasi-one-dimensional density of states in a waveguide, expressed in terms of the longitudinal velocities  $v_c = \hbar k_c/m$ . Inserting (16) and (17) into (13), we obtain

$$\tau' \approx \frac{2\pi\hbar}{\Lambda}(\rho_R^{\text{av}}(E) + \rho_R^{\text{f}}(E)). \tag{18}$$

This is the main result of this section. The first term on the r.h.s. of equation (18) represents the mean time spent in the cavity,  $\langle \tau \rangle = 2\pi\hbar/(\Lambda\Delta) = \tau_H/\Lambda$ , in agreement with Levinson's theorem (which holds irrespective of whether the underlying classical dynamics is chaotic or not; see, for example [17]). The second term is given by [16]

$$\tau^{\text{fl}}(E) \approx \frac{1}{\Lambda} \sum_{vm} T_v(E) A_{vm}(E) e^{ims_v(E)/\hbar - i\frac{\pi}{2}\mu_{vm}} \tag{19}$$

where the sum runs over the primitive periodic orbits  $v$  which do not leave the cavity and their  $m$ th repetitions. As usual, one needs from each periodic orbit  $v$  its period  $T_v$ , action  $s_v$ , Maslov index  $\mu_v$ , and the amplitudes  $A_v$  given in terms of the monodromy matrix  $M_v$  [16],

$$A_{vm} = \frac{1}{\sqrt{|\det(M_v^m - 1)|}}. \tag{20}$$

We base all considerations that follow on equation (19). One has to keep in mind that equation (19) is a semiclassical result, sharing most of the usual limitations of the standard trace formula for bound systems. It must be also emphasized that for the quantization of

the bound systems, one is free to seek a Poincaré map which includes all the periodic orbits (in our system this is the Birkhoff bounce map). Then, the r.h.s. of equation (11) is zero and the ‘delay time’ for this map is just the smooth Weyl density of states as discussed by Rouvinez and Smilansky [13]. The scattering map is predetermined in our case. This is a chaotic, singular, discontinuous classical map because of its exclusion of the repeller, which accounts for the fluctuations of the time delay. For this reason we cannot use the time delay calculated directly from the semiclassical scattering map, reworking the theory on the basis of the repeller.

### 3. Universal correlations in the quantum time delay

Here we apply the results of the last section to analyse the fluctuations of the time delay as a function of energy. We study the crossover regime in which time reversal symmetry is lost due to the presence of an external magnetic field  $B$ . Our derivation extends previous results by Bohigas *et al* [18] (closed systems at the crossover regime), by Eckhardt [19] (open systems with preserved time reversal symmetry) and by Fyodorov and Sommers [20] (open systems with broken time reversal symmetry by an Aharonov–Bohm flux line). Our presentation closely follows the discussion in [18].

A usual measure to characterize the time delay fluctuations is the correlation function

$$K_\tau(\varepsilon, B) = \langle \tau^{f_l}(E + \varepsilon, B) \tau^{f_l}(E, B) \rangle_E \quad (21)$$

where  $\langle \dots \rangle_E$  stands for energy average. The semiclassical approach to the calculation of this correlator begins by inserting (19) into (21) (the dependence of all quantities on  $B$  will now remain implicit to simplify the notation):

$$K_\tau(\varepsilon) = \frac{1}{\Lambda^2} \sum_{vmv'm'} \langle T_v(E + \varepsilon) T_{v'}(E) A_{vm}(E + \varepsilon) A_{v'm'}^*(E) \quad (22)$$

$$\times \exp[i m s_v(E + \varepsilon)/\hbar - i\pi \mu_{vm}/2 - i m' s_{v'}(E)/\hbar - i\pi \mu_{v'm'}/2] \rangle_E. \quad (23)$$

Here the average is taken over an energy interval  $\delta E$  which, to be meaningful, must be large as compared with the quantum scale  $\Delta$  in order to include many resonances. On the other hand, for practical purposes,  $\delta E$  must be small enough to allow for the use of classical perturbation theory. According to these considerations, the most important effect of varying the energy is on the actions, as they are measured in units of  $\hbar$ . We write

$$s(E + \varepsilon) \approx s(E) + \varepsilon T \quad A(E + \varepsilon) \approx A(E) \quad \text{and} \quad T(E + \varepsilon) \approx T(E). \quad (24)$$

To evaluate  $K_\tau(\varepsilon)$ , we use the ‘diagonal approximation’, neglecting contributions of pairs of orbits having distinct actions, as they cancel out upon averaging over energy. This approximation is accurate for orbits with periods shorter than some critical value, whose typical scale is the Heisenberg time  $\tau_H$ . We shall return to this point later.

For  $B = 0$  only two kinds of pairs of orbits survive: the pairs of identical orbits and the pairs of time-reversed partners. As the magnetic field  $B$  grows, the contribution to the correlation function of time-reversed partners gradually decreases. Keeping both contributions, we have

$$K_\tau(\varepsilon) = \frac{1}{\Lambda^2} \sum_{vm} \langle T_v^2 |A_{vm}|^2 e^{i\varepsilon m T_v/\hbar} [1 + e^{im\delta s_v/\hbar}] \rangle_E \quad (25)$$

where  $\delta s \equiv \delta s(E, B)$  stands for the action difference between a pair of time-reversed orbits. Next, we group orbits having periods in a small interval  $\delta t$  around  $t$  (containing

many orbits). This introduces a form of averaging procedure, defining smooth functions of  $t$ . We then integrate over all  $t$

$$K_\tau(\varepsilon) = \frac{1}{\Lambda^2} \int_{-\infty}^{\infty} dt |t| e^{i\varepsilon t/\hbar} \langle \langle |t| A^2 \rangle_t \langle [1 + e^{i\delta s/\hbar}] \rangle_t \rangle_E \quad (26)$$

where we discarded the multiple repetitions  $|m| \neq 1$ , which are exponentially negligible with respect to primitive orbits.

To evaluate the time average of the amplitudes  $A$  we use the sum rule for open chaotic systems [21, 22]:

$$\langle |t| A^2 \rangle_t = e^{-\gamma(E)|t|} \quad (27)$$

where  $\gamma(E)$ , the so-called escape rate, is  $\gamma = 1/\tau_{\text{dwell}}$ , with  $\tau_{\text{dwell}}$  defined in section 1. Equation (27) has a simple physical interpretation: as we increase the period, periodic orbits proliferate exponentially (with a rate given by the entropy) as their stability tend to decrease also exponentially (the rate given by the sum of positive Lyapunov exponents). For  $\gamma = 0$ , both effects cancel and one recovers the sum rule based on the uniformity principle for closed systems [23]. For open systems the disbalance between the entropy and Lyapunov exponents is just the escape rate resulting in (27) [24, 25].

The time average of the crossover term has been discussed in detail in [18]. Further theoretical arguments [26] and numerical evidence [27] suggest the exponential decay

$$\langle 1 + e^{i\delta s/\hbar} \rangle_t = 1 + e^{-B^2\kappa(E)|t|/\hbar^2}. \quad (28)$$

Here  $\kappa(E)$  is a purely classical quantity, which measures the rate of decay of the appropriate correlations in the chaotic system. For more details see [18]. Identifying  $\gamma(E)$  and  $\kappa(E)$  with their mean values (the energy averaging interval is small) we write

$$K_\tau(\varepsilon) = \int_{-\infty}^{\infty} dt |t| e^{i\varepsilon t/\hbar - \gamma|t|} (1 + e^{-B^2\kappa(E)|t|/\hbar^2}). \quad (29)$$

This integral is easily evaluated, resulting in

$$K_\tau(\omega) = \frac{1}{2} \langle \tau \rangle^2 \left\{ \frac{\Gamma^2 - \omega^2}{[\Gamma^2 + \omega^2]^2} + \frac{(\Gamma + y)^2 - \omega^2}{[(\Gamma + y)^2 + \omega^2]^2} \right\} \quad (30)$$

where, for the sake of future comparisons, we have defined

$$\omega = \pi\varepsilon/\Delta \quad \Gamma = \pi\gamma\hbar/\Delta \quad y = \frac{B^2\kappa\pi}{\hbar\Delta}. \quad (31)$$

When  $B = 0$ , we note that equation (30) reduces to the result obtained by Eckhardt [19] for the time reversal symmetric case. Alternatively, putting  $\gamma = 0$  we recover the results of [18] for the density–density correlator of the closed problem.

One of the most interesting properties of the semiclassical approximation to  $K_\tau(\omega)$  is that in this case we obtain an intrinsically more accurate result than the density–density correlators for closed systems, that have been extensively semiclassically studied. The reason is that the semiclassical approximation starts to fail for energy domains of the order of the mean level spacing  $\Delta$ , corresponding to times longer than  $\tau_H = 2\pi\hbar/\Delta$ , and such times are normally unimportant for the computation of  $K_\tau$ . The physics of the scattering problem provides us with a natural cut-off for the summation over periodic orbits, which is given by the typical escape time  $1/\gamma$ , explicit in (27). Since the semiclassical theory is only applicable if the waveguide  $L$  has many open channels,  $1/\gamma$  must be much smaller than the Heisenberg time  $\tau_H$ , corresponding to the regime of strongly overlapping resonances. Although we do not have a control over the magnitude of the accuracy, by increasing the number of open channels in actual systems,  $K_\tau(\omega)$  converges to the semiclassical approximation (30).



#### 4. Comparison with the statistical approach

The statistical properties of the Wigner–Smith time delay for chaotic systems have been also investigated by using the random matrix theory [28–31,20]. In analogy with the semiclassical approach, the statistical approach requires a decomposition of the Hilbert space into an asymptotic region  $L$  and a ‘complex’ scattering region  $R$ , following the notation of figure 1. As before, depending on the energy  $E$ , one will have  $\Lambda$  propagating channels in the pipe, labelled by  $c$ . By introducing arbitrary boundary conditions at  $x = D$  one can define a set of quasibound states  $|\mu\rangle$  in region  $R$  and a set of scattering states  $|\chi_c(E)\rangle$  in  $L$ . These states form a complete set  $Q + P = 1$ , with  $Q = \sum_\mu |\mu\rangle\langle\mu|$  and  $P = \sum_c \int dE |\chi_c(E)\rangle\langle\chi_c(E)|$ . Thus, the Hamiltonian is given by

$$H = QHQ + PHP + QHP + PHQ \equiv H_{QQ} + H_{PP} + H_{PQ} + H_{QP} \quad (32)$$

where  $H_{QQ}$  is interpreted as an ‘internal’ Hamiltonian, and  $H_{PQ}$  ( $H_{QP}$ ) are the couplings between the interior  $R$  and the channel region  $L$ . After some algebra, one can show [32, 17] that the resonant  $S$ -matrix can be written as

$$S = I - 2\pi i H_{PQ} \frac{1}{E - H_{QQ} + i\pi H_{QP} H_{PQ}} H_{QP} \quad (33)$$

in the absence of direct reactions, implying that  $H_{PP}$  is diagonal. The decomposition of the Hilbert space in projectors  $P$  and  $Q$  can, in principle, be employed for a large variety of problems, which makes equation (33) a very useful parametrization of the  $S$ -matrix. It follows that the Wigner–Smith time delay is given by [20]

$$\tau(E) = -\frac{2}{\Lambda} \text{Im} \text{Tr}(E - H_{QQ} + i\pi H_{QP} H_{PQ})^{-1}. \quad (34)$$

This expression is akin to the one obtained semiclassically (18), as it should be. Here  $\tau(E)$  is equated to the level density of the ‘closed’ system, defined by the operator  $Q$  (which is quite arbitrary) and smoothed by the coupling to the exterior world by the imaginary term in (34). Although conceptually similar to equation (18), it is not a simple task to arrive at the semiclassical expression starting from equation (34).

Since one expects signatures of chaos in scattering processes to be manifest for times much longer than the typical traversal time, we focus our attention only in the resonant part of  $S$ . This is the physical justification for neglecting direct (fast) reactions in equation (33). Moreover, the object which is responsible for classical chaos in scattering is the repeller, implying that chaos is a property of the ‘internal’ Hamiltonian. Therefore, in analogy with Bohigas’ conjecture [33] for closed systems, a statistical modelling of quantum chaos in open systems can be made by taking  $H_{QQ}$  as a member of an ensemble of random matrices [17]. For instance, for preserved time-reversal symmetry  $H_{QQ}$  belongs to the Gaussian orthogonal ensemble (GOE) and for broken time-reversal symmetry to the Gaussian unitary ensemble (GUE). This conjecture allows us to study universal fluctuations in scattering processes by calculating  $S$ -matrix correlation functions. Those are obtained by ensemble averaging, which is equivalent to an energy averaging based on the ergodic hypothesis.

In particular, the calculation of the two-point time delay correlation function  $K_\tau(\varepsilon, Y)$ , studied in the previous section, requires the evaluation of

$$K_\tau(\varepsilon, Y) = \frac{2}{\Lambda^2} \text{Re} \left\{ \overline{\text{Tr} g \left( E + \frac{\varepsilon}{2}, Y \right) \text{Tr} g^\dagger \left( E - \frac{\varepsilon}{2}, Y \right)} - \overline{\text{Tr} g(E, Y)^2} \right\} \quad (35)$$

where  $\overline{\mathcal{O}}$  denotes the ensemble average of  $\mathcal{O}$  and

$$g(E, Y) = (E - H_{QQ}(Y) + i\pi H_{QP} H_{PQ})^{-1} \quad (36)$$

with the variable  $Y$  parametrizing changes in the internal Hamiltonian. For instance, if  $Y$  stands for an external magnetic field  $B$ , one can study  $K_\tau(\varepsilon, Y)$  in the crossover regime between preserved and broken time reversal symmetry by choosing  $H_{QQ} = H^{\text{GOE}} + YH^{\text{GUE}}$ . The results are universal in terms of the scaled variable  $y = Y/Y_c$ , where  $Y_c$  is system specific.

From a technical point of view, the ensemble average in equation (35) implies a nontrivial calculation based on the supersymmetric technique developed by Efetov [34]. This technique was adapted to scattering problems by the Heidelberg group [35, 36], and is the starting point of all works that use the statistical approach to study the time delay correlation function [31, 37, 20] and related objects [38]. A discussion of the supersymmetric technique is beyond the scope of this paper and we refer the reader to the excellent introductory text in [39], and to the  $S$ -matrix review in [20].

Let us start with the simplest case,  $K_\tau(\omega, y \gg 1)$  for broken time-reversal symmetry, corresponding to taking  $H_{QQ}$  as a member of the GUE. The result, given as usual in terms of a double integral, can be found in [20]

$$K_\tau^{\text{GUE}}(\omega) = \frac{\langle \tau \rangle^2}{2} \int_{-1}^1 d\lambda \int_1^\infty d\lambda_1 \cos(\omega(\lambda_1 - \lambda)) \prod_{c=1}^\Lambda \left( \frac{2 + T_c(\lambda - 1)}{2 + T_c(\lambda_1 - 1)} \right) \quad (37)$$

where the transmission coefficient  $T_c$  gives the probability of an incoming wave at the channel  $c$  in the vicinity of  $x = D$  to enter the scattering region  $R$ . In order to compare with the semiclassical theory, one has to take  $T_c = 1$  for all channels  $c$ , since this theory does not take into account any barriers preventing perfect transmission. Thus, keeping the notation introduced in section 3 and identifying  $\langle \tau \rangle = 1/\gamma$ , the leading asymptotic term in powers of  $\Gamma^{-1}$  of equation (37) becomes

$$K_\tau^{\text{GUE}}(\omega) \approx \frac{\langle \tau \rangle^2}{2} \frac{\Gamma^2 - \omega^2}{[\Gamma^2 + \omega^2]^2} \quad (38)$$

in nice agreement with the semiclassical result. In figure 2 we present  $K_\tau^{\text{GUE}}$  as a function of the number of open channels  $\Lambda$ . As  $\Lambda$  increases, the agreement with the semiclassical theory becomes much better, as shown in the inset of figure 2. Even for relatively small  $\Lambda$ , the exact result does not differ significantly from the semiclassical one. This is explained by the fact that one can show that the next to leading-order correction is smaller by a factor  $\Gamma^2$ .

The other simple limit is the case where time reversal symmetry is present, corresponding to  $H_{QQ}$  taken as a member of the GOE. Here the result is [31]

$$K_\tau^{\text{GOE}}(\omega) = \frac{\langle \tau \rangle^2}{4} \int_0^1 d\lambda \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \mu(\lambda, \lambda_1, \lambda_2) (2\lambda + \lambda_1 + \lambda_2)^2 \quad (39)$$

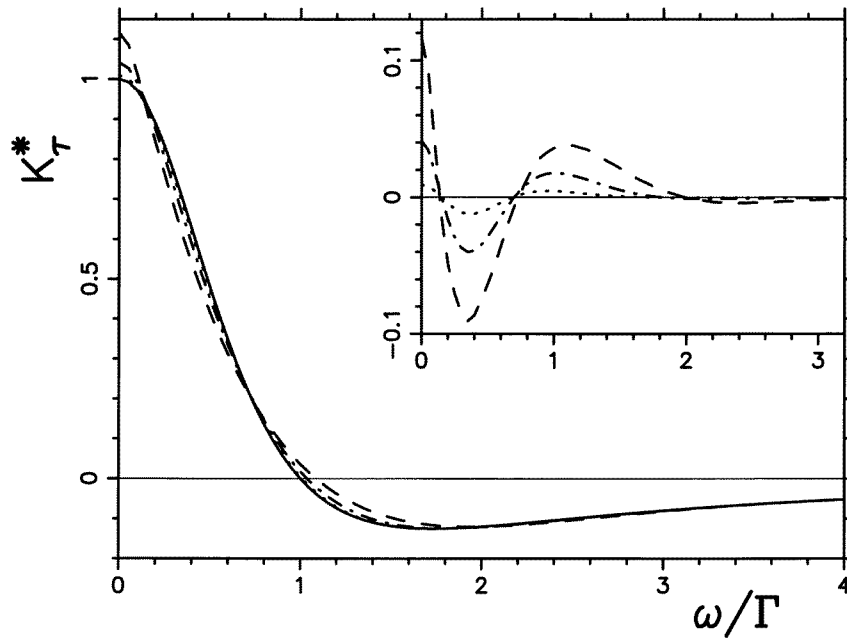
$$\times \cos(\omega(2\lambda + \lambda_1 + \lambda_2)) \prod_c \left( \frac{(1 - T_c \lambda)^2}{(1 + T_c \lambda_1)(1 + T_c \lambda_2)} \right)^{1/2} \quad (40)$$

with the measure  $\mu$  given by

$$\mu(\lambda, \lambda_1, \lambda_2) = \frac{(1 - \lambda)\lambda|\lambda_1 - \lambda_2|}{[(1 + \lambda_1)\lambda_1(1 + \lambda_2)\lambda_2]^{1/2}(\lambda + \lambda_1)^2(\lambda + \lambda_2)^2}. \quad (41)$$

In this case, even taking  $T_c = 1$  for all  $c$ 's the integral is still difficult. However, we can use a trick introduced by Efetov [34] or the asymptotic expansion proposed by Verbaarschot [36] to obtain

$$K_\tau^{\text{GOE}}(\omega) \approx \langle \tau \rangle^2 \frac{\Gamma^2 - \omega^2}{[\Gamma^2 + \omega^2]^2} \quad (42)$$



**Figure 2.** Comparison between the semiclassical (full lines) and the exact random matrix results for the time delay correlator (case where time-reversal symmetry is absent). We plot the normalized correlators  $K_T^* = 2\Gamma^2 K_T / \langle \tau \rangle^2$  versus the normalized energy  $\omega/\Gamma$ . Different line styles indicate the different number of open channels; broken, chained, and dotted correspond to  $\Lambda = 3, 5, 10$ , respectively. The inset shows the difference between random matrix results and semiclassics.

again in nice agreement with the semiclassical result. Here, higher-order corrections are smaller only by a factor  $\Gamma$ . This is manifest in figure 3, where one can see that the GUE case converges faster than the GOE to the semiclassical result.

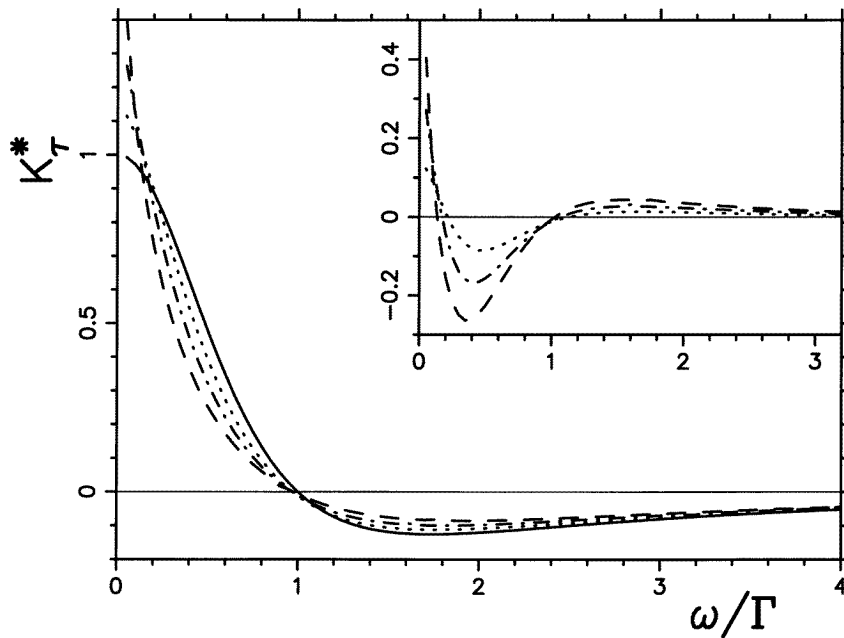
For the crossover regime the supersymmetric expressions become even more complicated. By performing a calculation similar to that done by Pluhař *et al* [40], Fyodorov *et al* [37] obtained a closed expression for  $K_T$ , first numerically studied in [41] as the ballistic limit of electronic mesoscopic transmission. The leading asymptotic term in inverse powers of  $\Gamma$  is

$$K_T(\omega, y) = \frac{\langle \tau \rangle^2}{2} \left( \frac{\Gamma^2 - \omega^2}{[\Gamma^2 + \omega^2]^2} + \frac{(\Gamma + y)^2 - \omega^2}{[(\Gamma + y)^2 + \omega^2]^2} \right) \quad (43)$$

identical to the semiclassical result equation (30).

The strength of the statistical method is in dealing with situations beyond the scope of the semiclassical theory, either by analysing situations where  $\Lambda$  is small, or by treating systems where the scattering waves have to overcome barriers. In the latter case, the semiclassical analysis needs some refinements. Equally or more problematic is the fact that when  $\sum T_c < 1$ , the resonances become isolated. This calls for a semiclassical theory that deals with times larger than  $\tau_H$ .

This is a good point to refocus the attention to one issue in the literature (see for instance [25]) that is often confusing. The ensemble average is performed for a fixed number of channels and many resonances between channel thresholds. The accuracy of the statistical approach increases as the number of resonances increase. With such a construction it is not



**Figure 3.** Comparison between the semiclassical (full lines) and the exact random matrix results for the time delay correlator (case of preserved time-reversal symmetry). We plot the normalized correlators  $K_T^* = \Gamma^2 K_T / \langle \tau \rangle^2$  versus the normalized energy  $\omega/\Gamma$ . Different line styles indicate the different number of open channels; broken, chain, and dotted correspond to  $\Lambda = 3, 5, 10$ , respectively. The inset shows the difference between random matrix results and semiclassics.

possible to use this theory to statistically analyse the  $S$ -matrix of a system like the three-disc problem, since between channel thresholds there are very few resonances, explaining the disagreement in [25]. However, since one takes the trace over the channels to obtain the Wigner–Smith time delay, provided  $\Lambda \gg 1$ , a small change in  $\Lambda$  seems not to affect  $K_T$ , as observed in [19].

## 5. Conclusions

In this paper we presented a semiclassical derivation of the formula connecting the Wigner–Smith time-delay  $\tau$  to the resonance density of the scattering region, corresponding to a chaotic system. We showed that  $\tau$  can be written as a sum over the periodic orbits inside the repeller. The physical interpretation of this relation is that the repeller is responsible for the time spent in the cavity by the scattering trajectories. An open trajectory that closely approximates a periodic one, can spend a long time in the scattering region. This dwell time essentially depends on the stability of the periodic trajectory that is approached. As a result, the typical classical dwell time depends on few bulk characteristics of the scattering system. The interesting achievement of the semiclassical theory is that it is possible to write scattering observables in terms of classical trajectories that never leave the system. Doing so, one avoids all problems inherent of a semiclassical formulation in terms of open trajectories.

One of the striking features of the semiclassical approximation for a scattering system is its accuracy. In distinction to the usual studies of density–density correlators in closed

systems, here one has no need to account for times of the order of  $\tau_H$ , the timescale where the semiclassical approach starts failing. Trajectories entering an open chaotic system cannot typically stay inside the scattering region for times much longer than  $\langle\tau\rangle$ . This fact provides us with a natural cut-off for any semiclassical summation formula, namely  $\langle\tau\rangle$  itself. If we assume that convergence is the only problem of the semiclassical formalism, systems with increasing numbers of open channels will be described by the semiclassical theory with increasing precision as compared with the exact theory.

Although we already know the exact statistical result for several correlators and distributions involving the Wigner–Smith time delay, such an approach, by construction, does not have information about nonuniversal quantities (such as  $\Delta$ ,  $Y_c$ , etc). Those are usually extracted from the experiment. The point of this paper is that this information is usually available from the classical dynamics and the semiclassical approach can always be adapted to give a method to compute the nonuniversal scaling factors. In summary, even if the semiclassical theory cannot compete in accuracy, it can be used as a complement to the statistical approach.

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